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## LETTER TO THE EDITOR

# Stability of the Parisi solution for the SK spin glass model at low temperatures close to the critical surface

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**Abstract.** In the framework of the Parisi solution of an infinite-ranged Ising spin glass we derive the equations which determine the free energy, the magnetisation and the function $q(x)$ for the general case $H \neq 0$ and $J_0 \neq 0$. We solved these equations for two cases: (i) $J_0 = 0, H \gg 1$; (ii) $J_0 \gg 1, H = 0$. An existence of massless modes is exactly proved.

Interest in the infinite-ranged model of a spin glass which was introduced by Sherrington and Kirkpatrick (1975, 1978) started with the physical idea that it may be solved exactly in the mean-field approximation and it is a good testing ground for this approach.

The replica-symmetry solution, proposed by Sherrington and Kirkpatrick (1975, 1978), becomes unstable at all temperatures $T$, magnetic fields ($H$) and mean value $J_0$ which lie below a critical surface in the space $(T, H, J_0)$ (de Almeida and Thouless 1978). A critical surface corresponds to the onset of the replica-symmetry breaking. The phase diagram $(T, H, J_0)$, expected for the SK model is depicted in figure 1 of Toulouse (1980). The attractive replica-symmetry breaking solution was proposed by Parisi (1979, 1980a, b, c). A stability analysis which has been performed by De Dominicis and Kondor (1983a, b) and Goltsev (1983a), who showed that the Parisi solution is stable for $T$ close to $T_c$. It has been shown that the free energy, the magnetisation ($m$) and the function $q(x)$ are governed by some self-consistent system of algebraic and differential equations (Goltsev 1983b). In the present paper we generalise these equations and prove exactly the existence of massless modes for the case $H \neq 0, J_0 \neq 0$. We have solved these equations for two cases: (i) $J_0 = 0, H \gg 1$; (ii) $J_0 \gg 1, H = 0$.

The Hamiltonian of the SK model of the Ising spin glass is

$$\mathcal{H} = -\sum_{ij} J_{ij} S_i S_j - H \sum_i S_i$$

for $N$ Ising spins $S_i$. The bond interactions $J_{ij}$ are taken as independent random variables with mean value $J_0/N$ and mean deviation $J/\sqrt{N}$. For simplicity of notation, we take the conventions $J = 1$, $k_B = 1$. The free energy per spin ($F$) is given by

$$\beta F = -\tfrac{1}{4}\beta^2 - \lim_{n\to 0} \frac{1}{n} \max\left[\frac{\beta^2}{4} \sum_{\alpha,\beta} Q^2_{\alpha\beta} - \tfrac{1}{2}\beta J_0 \sum_\alpha m^2_\alpha + \ln \operatorname{Tr} \exp\left(\tfrac{1}{2}\beta^2 \sum_{\alpha\neq\beta} Q_{\alpha\beta} S_\alpha S_\beta + \beta J_0 \sum_\alpha M_\alpha S_\alpha + \beta H \sum_\alpha S_\alpha\right) - 1\right] \quad (1)$$

 

(Sherrington and Kirkpatrick 1975, 1978, de Almeida and Thouless 1978). There is a non-trivial stationary point given by

$$m_\alpha = \langle S_\alpha \rangle = m, \qquad Q_{\alpha\beta} = \langle S_\alpha S_\beta \rangle \tag{2}$$

where for the matrix $Q_{\alpha\beta}$ we used the parametrisation proposed by Parisi (1979). Using a simple method proposed by Duplantier (1981) for the equations (1) and (2) we obtain

$$\beta F = -\tfrac{1}{4}\beta^2\left(1 + \int_0^1 q^2(x)\,\mathrm{d}x - 2q(1)\right) + \tfrac{1}{2}\beta J_0 m^2 - \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{(2\pi)^{1/2}} \mathrm{e}^{-z^2/2} f[0, \beta H + \beta J_0 m + \beta z (q(0))^{1/2}, \tag{3}$$

$$m = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{(2\pi)^{1/2}} \mathrm{e}^{-z^2/2} \varphi[0, \beta H + \beta J_0 m + \beta z (q(0))^{1/2}, \tag{4}$$

$$q(x) = \int_{-\infty}^{\infty} i \frac{\mathrm{d}z}{(2\pi)^{1/2}} \mathrm{e}^{-z^2/2} \psi_x [0, \beta H + \beta J_0 m + \beta z (q(0))^{1/2}], \tag{5}$$

where the functions $F(y, h)$ and $\varphi(y, h)$, $y \in [0, 1]$, and the function $\psi_x(y, h)$, $y \in [0, x]$, satisfy the equations

$$\partial f/\partial y = -\tfrac{1}{2}\beta^2 (\mathrm{d}q/\mathrm{d}y)[f'' + y(f')^2], \tag{6}$$

$$\partial \varphi/\partial y = -\tfrac{1}{2}\beta^2 (\mathrm{d}q/\mathrm{d}y)[\varphi'' + 2yf'\varphi'], \tag{7}$$

$$\partial \psi_x/\partial y = -\tfrac{1}{2}\beta^2 (\mathrm{d}q/\mathrm{d}y)[\psi_x'' + 2yf'\psi_x'], \tag{8}$$

where $f' \equiv \partial f/\partial h$, with the boundary conditions

$$f(1, h) = \ln(2 \cosh h), \qquad \varphi(1, h) = \tanh h, \qquad \psi_x(x, h) = \varphi^2(x, h). \tag{9}$$

For the case $J_0 = 0$ the equations (3) and (6) were obtained by Parisi (1980b) and the equations (4), (5), (7)–(9) were obtained by Goltsev (1983b) and de Almeida and Lage (1983).

The equations (3)–(9) determine the $F$, $m$ and $q(x)$ at all $T$, $H$ and $J_0$. For $T$ above the critical temperature $T_c(H, J_0)$ the replica symmetry is exact and the function $q(x)$ is constant. In this case equations (3)–(9) give the SK results.

Let us study the Parisi solution for the case $J_0 = 0$ and $H \gg 1$ when $T_c(H) \equiv T_c(H, 0) \simeq \tfrac{2}{3}(2/\pi)^{1/2} \exp(-H^2/2) \ll 1$. Omitting the details of our solution of (4)–(9) we present the next results for $T$ close to $T_c(H)$:

$$q(x) = \begin{cases} q(1), & x_1 \leq x < 1 \\ q(0) + a_1 x + \tfrac{1}{2} a_2 x^2 + \mathrm{O}(x^3), & x_0 \leq x < x_1 \\ q(0), & 0 < x < x_0 \end{cases} \tag{10}$$

$$\begin{aligned} q(1) &= q(H) + T_c^2(H)[\tfrac{11}{4}t - \tfrac{5}{7}t^2 + \mathrm{O}(t^3)] + \mathrm{O}(T_c^4(H)), \\ q(0) &= q_0(H) + T_c^2(H)[\tfrac{1}{4}t - \tfrac{5}{8}t^2 + \mathrm{O}(t^3)] + \mathrm{O}(T_c^4(H)), \\ q_0(H) &= 1 - \tfrac{3}{2}T_c^2(H) - \tfrac{3}{8}T_c^4(H)(H^2 - 1) + \mathrm{O}(T_c^6(H)), \\ x_1 &= \tfrac{1}{2} + \tfrac{1}{8}T_c^4(H)(H^2 - 1) + \tfrac{5}{28}t + \mathrm{O}(T_c^6(H)), \\ x_0 &= \tfrac{1}{2} + \tfrac{1}{8}T_c^4(H)(H^2 - 1) - \tfrac{5}{7}t + \mathrm{O}(T_c^6(H)), \end{aligned} \tag{11}$$

where $t = 1 - T/T_c(H)$. The coefficients $a_1$ and $a_2$ are given by

$$\beta^2 a_1 = \tfrac{14}{5} + \tfrac{86}{5}t + O(t^2),$$

$$\beta^2 a_1(x_1 - x_0) + \tfrac{1}{2}\beta^2 a_2(x_1 - x_0)^2 = \tfrac{5}{2}t + \tfrac{275}{56}t^2 + O(t^3).$$

Let us consider the case $H = 0$ and $J_0 \gg 1$ when a critical temperature $T_c(J_0) \equiv T_c(0, J_0) \simeq \frac{2}{3} - (2/\pi)^2 \exp(-J_0^2/2) \ll 1$. At $T > T_c(J_0)$ the replica-symmetric ferromagnetic phase is stable. At $T = T_c(J_0)$ there is a phase transition, associated with the breaking of the replica symmetry. For $T < T_c(J_0)$ a new broken-symmetry ferromagnetic phase is stable (de Almeida and Thouless 1978, Bray and Moore 1980). We have studied this phase for $T$ close to $T_c(J_0)(t = 1 - T/T_c(J_0) \ll 1)$. The function $q(x)$ up to terms of order $O(t^2)$ is determined by equations (10) and (11) with the magnetic field $H$ replaced by $J_0$. For $T < T_c(J_0)$ we obtain that the susceptibility $\chi$ is constant and is equal to

$$\chi = (2/\pi)^{1/2} e^{-J_0^2/2}[1 + (2/\pi)^{1/2} J_0 e^{-J_0^2/2}].$$

For the magnetisation $(m)$ we have

$$m = \begin{cases} m_0(J_0) - \frac{9}{8}T_c^3(J_0)(J_0 + J_0^{-1})t + O(t^2), & \text{for } T \geq T_c(J_0) \\ m_0(J_0) - \frac{9}{8}T_c^3(J_0)(J_0 + \frac{1}{6}J_0^{-1})t + O(t^2) & \text{for } T < T_c(J_0) \end{cases}$$

where $m_0(J_0)$ is equal to the magnetisation at $T = T_c(J_0)$:

$$m_0(J_0) = 1 - (2/\pi)^{1/2} \frac{1}{J_0} e^{-J^2/2}.$$

The following features should be noted: $m$ and $\chi$ have a singularity at the transition temperature $T_c(J_0)$.

Recently Sompolinsky (1981) showed that in the spin glass phase with the broken replica symmetry, there is a slow relaxation of the susceptibility from a non-equilibrium value $\chi_1 = \beta(1 - q(1))$ towards an equilibrium one $\chi_0$. At the same time the function $q$ decays from a finite value $(q(1))$ at $t_1$ to $q(0)$ at $t_0$. For the case $H = 0$ and $J_0 < 1$ $q(0)$ is equal to zero while for $J_0 > 1$ we obtain $q(0) \neq 0$, manifesting the incomplete decay of frozen correlation at the largest time scale.

Now we consider a stability of the Parisi solution. To perform the stability analysis, De Dominicis and Kondor (1983a, b) used a truncated model free energy, introduced by Parisi (1979). They showed that the stability matrix has three families of eigenvalues $(\{\lambda^{(1)}\}, \{\lambda^{(2)}\}, \{\lambda^{(3)}\})$. All these eigenvalues are greater or equal to $\lambda^{(3)}_{\min} = q(1) - t - q^2(1) = 0$ (see also Goltsev 1982a). For the free energy (1) the smallest eigenvalue of the third family is equal to

$$\lambda^{(3)}_{\min} = \tfrac{1}{2}\beta^3\left(T^2 - \int_{-\infty}^{\infty} \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \xi[0, \beta H + \beta J_0 m + \beta z(q(0))^{1/2}]\right) \tag{12}$$

(Goltsev 1982a, b), where the function $\xi(y, h)$, $y \in [0, 1]$, obeys equation (7) with the boundary condition $\xi(1, h) = \operatorname{sech}^4 h$. Unfortunately, $\lambda^{(1)}$ and $\lambda^{(2)}$ are unknown. However, it may be supposed that the inequality $\lambda^{(1)}, \lambda^{(2)} \geq \lambda^{(3)}_{\min}$ also satisfies (1) for the free energy.

Let us prove that $\lambda^{(3)}_{\min} = 0$. From equation (9) we have

$$(\partial/\partial x)\psi_x(x, h) = (\partial/\partial y)\psi_y(x, h)|_{y=x} + (\partial/\partial y)\psi_x(y, h)|_{y=x}$$

$$= 2\varphi(x, h)(\partial/\partial x)\varphi(x, h).$$

Using equations (7)–(8) yields

$$(\partial/\partial y)\psi_y(x,h)|_{y=x}=\beta^2(\mathrm{d}q/\mathrm{d}x)(\varphi'(x,h))^2$$

$(f'(y,h)=\varphi(y,h))$. From this equation and equations (9) we have

$$(\partial/\partial x)\psi_x(0,h)|_{x=1}=\beta^2(\mathrm{d}q/\mathrm{d}x)|_{x=1}\xi(0,h).$$

Using equation (5) we obtain

$$1=\beta^2\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(2\pi)^{1/2}}\,\mathrm{e}^{-z^2/2}\xi[0,\beta H+\beta J_0 m+\beta z(q(0))^{1/2}].$$

Therefore, $\lambda^{(3)}_{\min}=0$.

Using the equations (7)–(9) we calculate a value of $\partial\psi_y(x,h)/\partial y$ at $x=y=0$ and obtain the relation:

$$1=\beta^2\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(2\pi)^{1/2}}\,\mathrm{e}^{-z^2/2}(\varphi'(0,h))^2|_{h=\beta H+\beta J_0 m+\beta z(q(0))^{1/2}}.$$

In the case $H=0$ and $J_0<1(m=q(0)=0)$ this relation yields that the static (zero-field) susceptibility $\chi_0$ is equal to 1 at all $T<T_c=1$.

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